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BROADBAND DETECTION OF GAUSSIAN SIGNALS

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Submarine Combat Systems Directorate

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Report Documentation Page

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ABSTRACT

The well-known deflection coefficient for a stationary Gaussian signal in additive, independent, stationary Gaussian noise is derived. The optimal predetection, or Eckart, filter is also derived. It is shown that the least detectable Gaussian signal of specified power is proportional to the normalized square of the noise spectrum. It is also shown that maximally detectable Gaussian signals of specified power must have at least one spectral component of finite power and very narrow bandwidth.

ADMINISTRATIVE INFORMATION

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1. INTRODUCTION

The processing stream is depicted in Fig. 1. Pre-detection and post-detection filters are assumed linear and time invariant. The input $v(t)$ is assumed real valued, zero mean, Gaussian, and stationary, so $x(t)$ is also real valued, zero mean, Gaussian, and stationary. Due to the nonlinear square law device, $y(t)$ and $z(t)$ are real valued and stationary, but neither zero mean nor Gaussian. (If the input process is complex valued instead of real valued, as assumed here, the Eckart filter and corresponding deflection coefficient differ from the expressions given below by a constant.)

Assuming small signal to noise ratio, the variance under the null hypothesis

$$\{\mathbf{H}_0 : noise\ only\ is\ present\}$$

is approximately equal to the variance under the alternative hypothesis

$$\{\mathbf{H}_1 : signal\ plus\ noise\ is\ present\}.$$

The difference of the means of the output $z(t)$ under these hypotheses is divided by the standard deviation under the null hypothesis to obtain the so-called deflection coefficient, denoted by d. The receiver operator characteristic (ROC) curves of the detector are completely parameterized by d if Gaussian signal and noise statistical assumptions hold.

2. DETECTOR VARIANCE UNDER NOISE-ONLY HYPOTHESIS

The variance of the output $z(t)$ is computed first. Let $h_1(\tau)$ denote the impulse response function of the post-detection filter normalized so that

$$\int_{-\infty}^{\infty} h_1(\tau)\ d\tau = 1. \tag{2.1}$$

The output $z(t)$ is the convolution of the input $y(t)$ with $h_1(\tau)$, so that

$$z(t) = \int_{-\infty}^{\infty} y(t - \tau)h_1(\tau)\ d\tau. \tag{2.2}$$

Substituting (2.2) into the definition of the autocorrelation function of $z(t)$ gives

$$R_z(\tau) \equiv E[z(t)z(t + \tau)]$$

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$$= E\left[\int_{-\infty}^{\infty} y(t-\alpha)h_1(\alpha) \ d\alpha \ \int_{-\infty}^{\infty} y(t+\tau-\beta)h_1(\beta) \ d\beta\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[y(t-\alpha)y(t+\tau-\beta)]h_1(\alpha)h_1(\beta) \ d\alpha \ d\beta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_y(\tau-\beta+\alpha)h_1(\alpha)h_1(\beta) \ d\alpha \ d\beta, \qquad (2.3)$$

where the autocorrelation function of $y(t)$ is

$$\begin{aligned} R_y(\tau) &\equiv E[y(t)y(t+\tau)] \\ &= E[x^2(t)x^2(t+\tau)] \\ &= R_x^2(0) + 2R_x^2(\tau), \end{aligned} \qquad (2.4)$$

and where $R_x(\tau) \equiv E[x(t)x(t+\tau)]$ is the autocorrelation function of $x(t)$. The result (2.4) is found in many places; for example, Thomas[1] derives it from the fourth moment

$$\begin{aligned} E[x_1x_2x_3x_4] &= E[x_1x_2]E[x_3x_4] + E[x_1x_3]E[x_2x_4] \\ &\quad + E[x_1x_4]E[x_2x_3] - 2\ \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4 \end{aligned}$$

for general quadrivariate Gaussian random variables by setting $x_1 = x_2 = x(t)$ and $x_3 = x_4 = x(t+\tau)$ and using the fact that $x(t)$ is zero mean. Since $E[z^2] = R_z(0)$ and $E[z] = R_x(0)$, the variance of z is

$$\begin{aligned} Var[z] &= E[z^2] - E^2[z] \\ &= R_z(0) - R_x^2(0) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_y(-\beta+\alpha)h_1(\alpha)h_1(\beta) \ d\alpha \ d\beta - R_x^2(0) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{R_x^2(0) + 2R_x^2(-\beta+\alpha)\right\} h_1(\alpha)h_1(\beta) \ d\alpha \ d\beta - R_x^2(0) \\ &= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_x^2(-\beta+\alpha)h_1(\alpha)h_1(\beta) \ d\alpha \ d\beta, \qquad (2.5) \end{aligned}$$

where the normalization (2.1) is used in the last step. The result (2.5) is easily evaluated for any specified post-detection filter.

The simplest low-pass post-detection filter satisfying the normalization (2.1) is $h_1(t) = 1/T$ for $|t| \le T/2$ and $h_1(t) = 0$ for $|t| \ge T/2$. It is assumed that the averaging time T is large compared to the correlation time of $x(t)$. Substituting

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into (2.5) gives

$$
\begin{aligned}
Var[z] &= \frac{2}{T} \int_{-T}^{T} R_x^2(\tau) \left(1 - \frac{|\tau|}{T} \right) \, d\tau \\
&\cong \frac{2}{T} \int_{-\infty}^{\infty} R_x^2(\tau) \, d\tau \\
&= \frac{1}{\pi T} \int_{-\infty}^{\infty} S_x^2(\omega) \, d\omega,
\end{aligned}
\tag{2.6}
$$

where the power spectrum $S_x(\omega)$ of $x(t)$ is the Fourier transform

$$
S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-i\omega\tau} \, d\tau.
\tag{2.7}
$$

The last equation in (2.6) is Parseval's theorem for (2.7), namely,

$$
\int_{-\infty}^{\infty} R_x^2(\tau) \, d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x^2(\omega) \, d\omega.
$$

Denoting the impulse response and transfer functions of the pre-detection filter by $h_0(\tau)$ and $H_0(\omega)$, respectively, gives

$$
S_x(\omega) = |H_0(\omega)|^2 S_v(\omega).
\tag{2.8}
$$

Substituting (2.8) into (2.6) gives

$$
Var[z] = \frac{1}{\pi T} \int_{-\infty}^{\infty} |H_0(\omega)|^4 S_v^2(\omega) \, d\omega.
\tag{2.9}
$$

The standard deviation of $z(t)$ is the square root of (2.9).

3. DEFLECTION COEFFICIENT

The difference in the means of $z(t)$ under hypotheses $\mathbf{H_0}$ and $\mathbf{H_1}$ is now computed. Denote the correlation functions and power spectra of the input $v(t)$ under hypotheses $\mathbf{H_0}$ and $\mathbf{H_1}$ by $R_v^N(\tau)$ and $S_v^N(\omega)$, and by $R_v^{S+N}(\tau)$ and $S_v^{S+N}(\omega)$, respectively. Signal and noise are statistically independent, so their power spectra add; that is,

$$
S_v^{S+N}(\omega) = S_v^S(\omega) + S_v^N(\omega),
\tag{3.1}
$$

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where $S_v^s(\omega)$ denotes the spectrum of the signal when signal only is present in $v(t)$. The pre-detection filter is linear and time invariant, so the difference in the mean of the output $z(t)$ under hypotheses $\mathbf{H}_0$ and $\mathbf{H}_1$ is

$$
\begin{aligned}
\Delta\mu &\equiv E[z|H_1] - E[z|H_0] \\
&= R_x^{S+N}(0) - R_x^N(0) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}|H_0(\omega)|^2 S_v^{S+N}(\omega)\,d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty}|H_0(\omega)|^2 S_v^N(\omega)\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}|H_0(\omega)|^2 S_v^s(\omega)\,d\omega,
\end{aligned}
\tag{3.2}
$$

where in the last step (3.1) has been used.

The deflection coefficient for Gaussian random variables with identical variances is defined by

$$
d = \frac{\Delta\mu}{\sigma},
\tag{3.3}
$$

where σ denotes the common standard deviation. Because of the low signal to noise ratio (SNR) assumption, the standard deviations of $z(t)$ under $\mathbf{H}_0$ and $\mathbf{H}_1$ are approximately equal. Substituting (3.2) and (2.9) into (3.3) gives

$$
d = \frac{\frac{1}{2}\sqrt{\frac{T}{\pi}}\int_{-\infty}^{\infty}|H_0(\omega)|^2 S_v^s(\omega)\,d\omega}{\left[\int_{-\infty}^{\infty}|H_0(\omega)|^4\left[S_v^N(\omega)\right]^2\,d\omega\right]^{\frac{1}{2}}}.
\tag{3.4}
$$

If the signal spectrum is unknown, the pre-detection filter is sometimes omitted, so that

$$
d = \frac{\frac{1}{2}\sqrt{\frac{T}{\pi}}\int_{-\infty}^{\infty}S_v^s(\omega)\,d\omega}{\left[\int_{-\infty}^{\infty}\left[S_v^N(\omega)\right]^2\,d\omega\right]^{\frac{1}{2}}},
\tag{3.5}
$$

which is the expression used for the conventional processor.

4. ECKART FILTER

The deflection (3.4) is easily maximized in any specified frequency band, say Ψ, for which the noise spectrum is bounded away from zero; that is, $S_v^N(\omega) \geq \varepsilon > 0$ for $\omega \in \Psi$. For $\omega \in \Psi$, write

$$
|H_0(\omega)|^2 S_v^s(\omega) = \left[|H_0(\omega)|^2 S_v^N(\omega)\right]\left[\frac{S_v^s(\omega)}{S_v^N(\omega)}\right].
$$

4

From the Cauchy-Schwartz inequality,

$$\int_\Psi |H_0(\omega)|^2 \, S_v^S(\omega) \, d\omega \leq \left[\int_\Psi \left[|H_0(\omega)|^2 \, S_v^N(\omega)\right]^2 \, d\omega\right]^{\frac{1}{2}} \left[\int_\Psi \left[\frac{S_v^S(\omega)}{S_v^N(\omega)}\right]^2 \, d\omega\right]^{\frac{1}{2}}. \tag{4.1}$$

Substituting (4.1) into (3.4) gives the inequality

$$d \leq \left[\int_\Psi \left[\frac{S_v^S(\omega)}{S_v^N(\omega)}\right]^2 \, d\omega\right]^{\frac{1}{2}}. \tag{4.2}$$

The inequality (4.2) holds for all $H_0(\omega)$. The upper bound is independent of $H_0(\omega)$, so equality holds in (4.2) if and only if equality holds in (4.1). The inequality (4.1) is an equality if and only if, for some (real) proportionality constant κ,

$$|H_0(\omega)|^2 \, S_v^N(\omega) = \kappa \, \frac{S_v^S(\omega)}{S_v^N(\omega)} \tag{4.3}$$

for all $\omega \in \Psi$. The deflection d is invariant to the scale of $H_0(\omega)$, so κ is set equal to one. Solving (4.3) for $|H_0(\omega)|^2$, and relabeling it $H_{Eckart}(\omega)$, gives

$$|H_{Eckart}(\omega)|^2 = \frac{S_v^S(\omega)}{[S_v^N(\omega)]^2}. \tag{4.4}$$

The optimum filter (4.4) is often called the Eckart filter, after its discoverer.[2]

Substituting the optimum filter (4.4) into (3.4) gives

$$d_{Eckart} = \frac{1}{2}\left[\frac{T}{\pi} \int_\Psi \left[\frac{S_v^S(\omega)}{S_v^N(\omega)}\right]^2 \, d\omega\right]^{\frac{1}{2}}. \tag{4.5}$$

The last result shows that $(d_{Eckart})^2$ is directly proportional to the averaging time T of the post-detection filter and the integral of the square of the SNR.

Scholium. Eckart's original derivation[2] does not use the Cauchy-Schwartz inequality. Instead, he writes a candidate expression for the optimum filter, and then proves that this expression is indeed optimal. He gives no indication of how he initially obtained the correct filter.

5. EXTREMAL SIGNALS OF SPECIFIED POWER

5.1. Least Detectable Signal

Given the noise spectrum $S_v^N(\omega)$ and a specified signal power, say S_{power}, the least detectable signal $S_{opt}(\omega)$ is the signal for which the deflection coefficient d_{Eckart} is a minimum. Let $\alpha \geq 0$ denote the SNR. Then $S_{opt}(\omega)$ is the solution of the following calculus of variations problem:

$$\min_{S(\omega)} \quad \int_\Psi \left[\frac{S(\omega)}{S_v^N(\omega)} \right]^2 \, d\omega \tag{5.1}$$

subject to the SNR constraint

$$\frac{S_{power}}{\int_\Psi S_v^N(\omega) \, d\omega} = \frac{\int_\Psi S(\omega) \, d\omega}{\int_\Psi S_v^N(\omega) \, d\omega} = \alpha. \tag{5.2}$$

Let $S(\omega) = f(\omega) \, S_v^N(\omega)$, where $f(\omega) \geq 0$, and let

$$G(\omega) = \frac{S_v^N(\omega)}{\int_\Psi S_v^N(\omega) \, d\omega} \geq 0. \tag{5.3}$$

The problem (5.1)–(5.2) written in terms of $f(\omega)$ and $G(\omega)$ takes the form

$$\min_{f(\omega)} \quad \int_\Psi f^2(\omega) \, d\omega \tag{5.4}$$

subject to

$$\int_\Psi f(\omega)G(\omega) \, d\omega = \alpha. \tag{5.5}$$

It is shown below that the solution to (5.4)–(5.5) is non-negative, so the constraint $f(\omega) \geq 0$ is ignored here without loss of generality. The appropriate Lagrangian function for (5.4)–(5.5) is

$$\mathcal{L}(f) = \frac{1}{2} \int_\Psi f^2(\omega) \, d\omega - \lambda \left(\int_\Psi f(\omega)G(\omega) \, d\omega - \alpha \right). \tag{5.6}$$

Let $\delta f(\omega)$ denote a fixed variation of $f(\omega)$, and let ϵ be any real number. Then the variation of (5.6) is

$$\mathcal{L}(f + \epsilon\delta f) = \frac{1}{2} \int_\Psi [f(\omega) + \epsilon\delta f(\omega)]^2 \, d\omega - \lambda \left(\int_\Psi [f(\omega) + \epsilon\delta f(\omega)] \, G(\omega) \, d\omega - \alpha \right). \tag{5.7}$$

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The derivative of $\mathcal{L}(f + \epsilon \delta f)$ with respect to ϵ must be zero at $\epsilon = 0$; hence,

$$\left. \frac{\partial \mathcal{L}(f + \epsilon \delta f)}{\partial \epsilon} \right|_{\epsilon=0} = \int_{\Psi} [f(\omega) - \lambda G(\omega)] \; \delta f(\omega) \; d\omega = 0, \qquad (5.8)$$

for all variations $\delta f(\omega)$. For (5.8) to hold for all $\delta f(\omega)$, it must be the case that $f(\omega) - \lambda G(\omega) \equiv 0$. Substituting $f(\omega) \equiv \lambda G(\omega)$ into the constraint (5.5) and integrating over ω gives

$$\alpha = \int_{\Psi} [\lambda G(\omega)] \, G(\omega) \; d\omega = \lambda \int_{\Psi} G^2(\omega) \; d\omega.$$

Solving the last equation for λ and substituting into $f(\omega) \equiv \lambda G(\omega)$ gives the optimal solution

$$f_{opt}(\omega) = \frac{\alpha G(\omega)}{\int_{\Psi} G^2(\omega) \; d\omega} \geq 0. \qquad (5.9)$$

Because

$$\left. \frac{\partial^2 \mathcal{L}(f + \epsilon \delta f)}{\partial \epsilon^2} \right|_{\epsilon=0} = \int_{\Psi} [\delta f(\omega)]^2 \; d\omega > 0$$

for all nonzero variations $\delta f(\omega)$, the sufficient conditions for the solution (5.9) to be a minimum hold; thus, $f_{opt}(\omega)$ is the unique global minimum of (5.4)–(5.5). In the original problem (5.1)–(5.2), the signal that minimizes the deflection coefficient d_{Eckart} is

$$\begin{aligned} S_{opt}(\omega) &= \frac{\alpha \int_{\Psi} S_v^N(\omega) \; d\omega}{\int_{\Psi} [S_v^N(\omega)]^2 \; d\omega} \; [S_v^N(\omega)]^2 \\ &= S_{power} \frac{[S_v^N(\omega)]^2}{\int_{\Psi} [S_v^N(\omega)]^2 \; d\omega}. \end{aligned} \qquad (5.10)$$

In words, the spectrum of the least detectable signal is directly proportional to the normalized square of the noise spectrum, and the proportionality constant is the specified signal power.

5.2. Maximally Detectable Signals

If noise power is finite and its spectrum is continuous and bounded on the band Ψ, then the deflection coefficient d_{Eckart} can be made arbitrarily large for any specified signal power $S_{power} > 0$. For example, if the signal spectrum is

$$S_\epsilon(\omega) = \begin{cases} \epsilon^{-1} S_{power}, & a \leq \omega \leq a + \epsilon, \\ 0, & \text{otherwise}, \end{cases}$$

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where a and $\epsilon > 0$ are such that the interval $[a, a + \epsilon]$ is a subset of Ψ, then

$$\int_\Psi S_\epsilon(\omega) \, d\omega = \int_a^{a+\epsilon} S_\epsilon(\omega) \, d\omega = S_{power}$$

and

$$\int_\Psi \left[\frac{S_\epsilon(\omega)}{S_v^N(\omega)} \right]^2 \, d\omega = \frac{S_{power}^2}{\epsilon^2} \int_a^{a+\epsilon} [S_v^N(\omega)]^{-2} \, d\omega. \qquad (5.11)$$

By the mean value theorem of calculus, there exists a point $\omega_\epsilon \in [a, a + \epsilon]$ such that

$$\int_a^{a+\epsilon} [S_v^N(\omega)]^{-2} \, d\omega = \epsilon \, [S_v^N(\omega_\epsilon)]^{-2} . \qquad (5.12)$$

Substituting (5.12) into (5.11) gives

$$\int_\Psi \left[\frac{S_\epsilon(\omega)}{S_v^N(\omega)} \right]^2 \, d\omega = \frac{1}{\epsilon} \frac{S_{power}^2}{[S_v^N(\omega_\epsilon)]^2} \geq \frac{1}{\epsilon} \left[\frac{S_{power}}{\max\limits_{\omega \in \Psi} S_v^N(\omega)} \right]^2 . \qquad (5.13)$$

By assumption, the noise spectrum is bounded on the band Ψ, so the maximum in (5.13) is finite. Thus, d_{Eckart} is made arbitrarily large by choosing ϵ sufficiently small. In words, the most detectable signals are those whose spectra contain at least one component of finite power and very narrow bandwidth.

6. SUMMARY

The deflection coefficient d for a Gaussian signal in additive, independent Gaussian noise is derived. The optimal pre-detection, or Eckart, filter is also derived. The signal of specified power that is least detectable by the optimal processor is directly proportional to the normalized square of the noise spectrum, and it is obtained using a calculus of variations argument. It is also shown that the most detectable signals of specified power have at least one spectral component of finite power and very narrow bandwidth.

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[1] J. B. Thomas, *An Introduction to Statistical Communication Theory* (Wiley & Sons, New York, 1969), Equation (6.4–19).

[2] C. Eckart, "Optimal Rectifier Systems for the Detection of Steady Signals," Scripps Institution of Oceanography, La Jolla, California, SIO Reference 52-11, 4 March 1952.
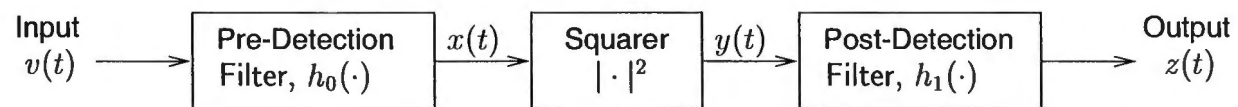
Figure 1. Processing Stream

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